IK2219/IK3506
Homework #3: Queuing Theory from *Palm* Viewpoint

Due on 23:55, September 28th (Sunday), 2014

Jeong-woo Cho

Rules

Each student (no grouping) should carry out this homework. Although you are encouraged to discuss with each other, you have to write programming codes and solutions **in your own words**. You can hand in solutions either in a **single** Acrobat PDF file or Microsoft Word file. The length of the submission should not be more than 5 pages (font size: 10-12, paper size: A4/Letter).

If you **miss** the deadline specified on the front page of this document and hand in your solutions within exactly one week after the deadline, **30% of the total points will be taken off**, regardless of your actual score. Solutions handed in more than one week after the deadline will not be graded. Most importantly, if your solutions are incorrect, you are highly likely to receive 0 points for the corresponding questions. In this light, you should double-check if **your solutions are unequivocally correct**.

Note again that you are allowed to use only built-in functions in MATLAB and its toolboxes, *e.g.*, Statistics Toolbox. It is not allowed to use SIMULINK.

Instructions

Some laws of queuing theory (Chapter 8) are so fundamental that they well deserve to be remembered. The hard truth is that understanding a theory is closely related to memorizing. However, it is important to memorize the underlying *implications* of them, rather than the equations or expressions, *i.e.*, their appearances. If you remember only their appearances, you are not likely to be able to *conjure* them *up* when looking for appropriate theoretical results for a given problem in your research because it is your understanding of the implications of a theory which will connect your process of thinking to the the theory suitable for your problem.

In order to remember the implications of important results in queuing theory, we will conduct some simulations to verify them in this homework. Also, this homework will enlighten you on the central role of *Palm viewpoint* in queuing theory. In other words, we will understand why it is much easier to comprehend queueing theory through Palm calculus, which we (will) study in Chapter 7. Throughout this homework, we will use an M/GI/1 queue for simulations. Let us recall that the service discipline in Kendall's notation is, by default, FIFO.

Problem 1: PASTA

The most fundamental property of queueing theory, PASTA (**P**oisson **A**rrivals **S**ee **T**ime **A**verages), and its implication may sound a little bit pedantic at the first time when you hear it. Although this basic property will be treated in Chapter 7 (Palm Calculus), you are strongly recommended to solve this problem and *feel* the PASTA property before attending the corresponding lecture. To put it simply, the basic implication of PASTA is as follows:

"If a Poisson arrival process samples a queuing system just <u>before</u> its arrival to the system, the sampled probability (or distribution) is equal to the one sampled at arbitrary time."

where a Poisson arrival process (precisely speaking, homogeneous Poisson) can be defined as a point process whose interarrival times are exponentially distributed. In fact, this property is, in some sense, too self-evident to be called a 'property' because the number of events in any very small (precisely speaking, infinitesimal) interval Δt of a Poisson process with its associated parameter λ is always $\Delta t \times \lambda$ for all t, which in fact means that an event of a Poisson process occurs at an **arbitrary** time.

In order to experience PASTA property, let us make a simulation code for M/GI/1 queue (the maximum number of customers allowed in the system K is ∞). The arrival process is Poisson with its associated parameter $\lambda = 0.0333333$/sec, or equivalently, the interarrival times of the arrival process is exponentially distributed with mean $1/\lambda = 30$ sec. The service process has Weibullian service times S whose probability density function is:

$$f_S(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{(-x/a)^b}, \quad x > 0.$$

where $b = 1/2 = 0.5$ and $a = 10$ sec. Then it can be easily seen that the mean and standard deviation of service time are $\bar{S} = a \times \Gamma(1 + 1/b) = 20$ sec and $\sigma_S = 44.72135955$ sec, respectively. The corresponding *coefficient of variation* is $\sigma_S/\bar{S} = 2.236067978$. The simulation should terminate at (simulated) time $T_s = 365 \times 86400\ sec = 1$ year.

(a)

As discussed in Chapter 8.3.2, the mean value of number of customers in waiting room (excluding the customer who is being served), sampled at **arbitrary time** is:

$$\bar{N}_w = \frac{\rho^2 \kappa}{1-\rho}, \quad \kappa = \frac{1}{2}\left(1 + \frac{\sigma_S^2}{\bar{S}^2}\right).$$

Plugging $\rho = \bar{S} \times \lambda = 2/3$, $\bar{S} = 20$ sec and $\sigma_S = 44.72135955$ sec into the above expression yields:

$$\bar{N}_w = 4 \text{ customers.}$$

Write a MATLAB program code to measure the number of customers in waiting room sampled **just before the arrival of each customer** to the queue, where we denote the mean of the measured values by $\hat{N}_w$. Show the histogram of the measured value by using `hist()` with 20 bins (`nbins=20`). Compare $\hat{N}_w$ with $\bar{N}_w$ to check if PASTA property holds in this case.

(b)

Likewise, the mean value of number of customers in system (including the customer who is being served) sampled at **arbitrary time**, which is given by:

$$\bar{N} = \frac{\rho^2 \kappa}{1-\rho} + \rho = 4.66666667 \text{ customers.}$$

Write a MATLAB program code to measure the number of customers in system sampled **just before the arrival of each customer** to the queue, where we denote the mean of the measured values by $\hat{N}$. Show the histogram of the measured value with 20 bins. Compare $\hat{N}$ with $\bar{N}$ to check if PASTA property holds in this case.

Do **not** attach your MATLAB program here. In the last problem of this homework, you will be asked to attach a *single* MATLAB program code which generates solutions to all the problems in this homework. If PASTA has been engraved on your mind, you are ready to tackle out the next problem.

Problem 2: Feller's Paradox

Why do you *feel* that you wait longer than others when shopping?

Prior to the formal treatment of Feller's paradox in Chapter 7 (Palm Calculus), we want to feel in this homework how the reality (which is stated by Feller's paradox) can be slightly different from what your intuition tells to you. That is probably why it is called a 'paradox'.

Problem 2: Feller's Paradox continued on next page...

Let us recall from the previous problem that $\bar{N}_w$ is the mean of customers in the waiting room (excluding the customer who is being served) sampled at arbitrary time, which is equal to the the one sampled just before the arrival of each customer, due to PASTA property. **Applying** this relation, we can easily see that, whenever a customer arrives at the queue, there are on the average $\bar{N}_w$ customers in the waiting room. Based on this observation, it is very much tempting to conjecture that the mean waiting time should be:

$$\begin{aligned} &\text{waiting time (of the arriving customer) generated by customers in the waiting room} && (1)\\ +\ &\text{waiting time (of the arriving customer) generated by customers being served} && (2) \end{aligned}$$

where (1) must be $\bar{N}_w \times \bar{S}$. Moreover, (2) **seems** to be $\rho \times \bar{S}/2$ because the utilization factor $\rho = \lambda \times \bar{S}$ is the probability that the server is busy (If you cannot understand it, take at look at ***Utilization Law*** in Chapter 8.2.2.) and it is in line with our intuition that the the mean residual service time should be one half of $\bar{S}$. Plugging the parameters, $\rho = \bar{S} \times \lambda = 2/3$ and $\bar{S} = 20$ sec, into our conjectured expression yields:

$$\hat{W} = \bar{N}_w \bar{S} + \frac{1}{2} \cdot \rho \bar{S} = 4 \times 20 + \frac{1}{2} \times \frac{2}{3} \times 20 = 86.66666667 \text{ sec}$$

Since the above example can be taken as an analogy of queues of customers waiting for cashiers in ICA and COOP, $\hat{W}$ corresponds to what our (wrong!) **intuition** tells us about the expected waiting time when we are shopping.

On the contrary, if we anatomize the formula for mean waiting time in Chapter 8.3.2, we have the following terms:

$$\bar{W} = \frac{\rho \cdot \bar{S} \cdot \kappa}{1-\rho} = \frac{\rho \bar{S} \kappa - (1-\rho)\rho \bar{S} \kappa}{1-\rho} + \rho \bar{S} \kappa = \frac{\rho^2 \bar{S} \kappa}{1-\rho} + \rho \bar{S} \kappa = \bar{N}_w \bar{S} + \kappa \cdot \rho \bar{S}.$$

Plugging the parameters into the above expression yields

$$\bar{W} = \bar{N}_w \bar{S} + \rho \bar{S} \kappa = 4 \times 20 + \frac{2}{3} \times 20 \times \frac{1}{2} \left(1 + \sqrt{5}^2\right) = 120 \text{ sec}$$

which is much longer than 86.66666667 sec. In the same vein, $\bar{W}$ corresponds to what **the grim reality** tells us about the expected waiting time in shopping centers.

(a)

Measure the waiting time of each customer (excluding the service time). Show the histogram of the measured value with 50 bins. Compare the mean waiting time measured through the simulation with $\bar{W} = 120$ sec and $\hat{W} = 86.66666667$ sec. Among $\bar{W}$ and $\hat{W}$, which one is closer to the measured mean waiting time?

(b)

If we liken this queuing system to the customer queues in shopping centers such as ICA, COOP and LIDL, the gap between $\hat{W}$ and $\bar{W}$ might be able to explain us why we *feel* that we usually wait a little bit longer than what we would expect when we queue up for cashiers in shopping centers. If there are *four* ($\bar{N}_w = 4$) customers on the average in front of us, we would guess that we need to wait $\bar{N}_w \bar{S} + \frac{1}{2}\rho \bar{S}$. But the simulation result alludes to our wrong intuition.

In fact, the two expressions, $\hat{W} = \bar{N}_w \bar{S} + \frac{1}{2}\rho \bar{S}$ (intuitive one) and $\bar{W} = \frac{\rho \bar{S} \kappa}{1-\rho}$ (formula in the textbook) satisfy the following inequality for any probability distribution of S (service time):

$$\bar{W} \geq \hat{W}$$

which means the actual waiting time is statistically longer than what our intuition tells us. Prove this inequality. When does the equality $\bar{W} = \hat{W}$ hold?

(c)

Attach here a single MATLAB program code which generates all solutions (in particular, figures) to all problems in this homework. You don't need to commentate the code.

Before finishing this homework, make sure that there are three figures in total in your submission. Lastly, you would better start the next homework as soon as possible because ***the complexity of its simulation is considerably higher***.